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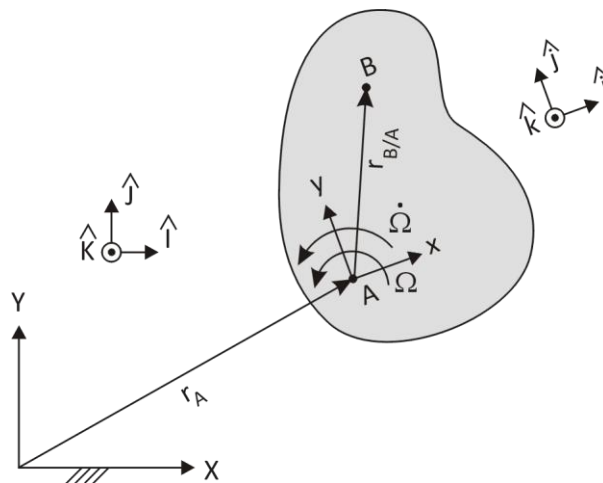
Rotating reference frame
and the five-term acceleration equation

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In Dynamics it is often useful to describe motion relative to a coordinate frame that moves and that rotates—that is, not by referring to a inertial or Newtonian frame that is fixed to the Earth. This can happen if it's needed to describe or analyze motion of components of a vehicle, that itself is moving in a fixed frame. Here we develop the so-called *five-term acceleration equation* in two dimensions, for the purpose of easier visualization. What is developed here can easily be extended to 3-D. This is one of the very nice results in Dynamics, because different types of acceleration come from this derivation, and the derivation itself simply involves repeated use of the product rule when differentiating vectors.

We start by describing the motion to be analyzed, which involves every possibility of motion when there are two frames—one stationary and one moving and rotating—used to describe the motion. The figure below shows the scenario.

xy-frame moving and rotating inside XY-frame

A conventional 2-D kinematic slab is shown. It is moving in a fixed reference frame, denoted with capital letters. On the slab is fixed a small-letter frame with A as its origin. This frame moves with the slab. Besides moving around in XY-space, the slab is also rotating at an angular speed Ω , which is also changing at a rate $\dot{\Omega}$. On the moving, rotating slab another point B has a motion of its own described

in the little-letter coordinate system. Thus B moves relative to A, and this motion is described relative to the slab.

Position

We are interested in the motion of B in the *XY*-frame, since the absolute motion is what Newton's Second Law applies to. We start with

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

To see how $\vec{r}_B$ changes with time, it is useful to break it up into magnitudes and directions using unit vectors. Thus

$$\vec{r}_B = r_{AX}\hat{I} + r_{AY}\hat{J} + r_{B/A-x}\,\hat{i} + r_{B/A-y}\,\hat{j}$$

Since $\vec{r}_{B/A}$ is a vector described in the little-letter coordinate system, it is more natural to express it in terms of $\hat{i}$ and $\hat{j}$ components.

Velocity

By definition

$$\vec{v}_B = \dot{\vec{r}}_B$$

We use the product rule to differentiate $\vec{r}_B$. So

$$\vec{v}_B = \dot{\vec{r}}_B = \dot{r}_{AX}\hat{I} + r_{AX}\dot{\hat{I}} + \dot{r}_{AY}\hat{J} + r_{AY}\dot{\hat{J}} + \dot{r}_{B/A-x}\,\hat{i} + r_{B/A-x}\dot{\hat{i}} + \dot{r}_{B/A-y}\,\hat{j} + r_{B/A-y}\dot{\hat{j}}$$

Since $\hat{I}$ and $\hat{J}$ are fixed, $\dot{\hat{I}}$ and $\dot{\hat{J}}$ are both 0. So

$$\vec{v}_B = \dot{r}_{AX}\hat{I} + \dot{r}_{AY}\hat{J} + \dot{r}_{B/A-x}\,\hat{i} + r_{B/A-x}\dot{\hat{i}} + \dot{r}_{B/A-y}\,\hat{j} + r_{B/A-y}\dot{\hat{j}}$$

All $\dot{r}$ are changes in the magnitudes of the corresponding r vectors, which are speed-ups or slow-downs. They are velocities, by definition. Thus

$$\vec{v}_B = v_{AX}\hat{I} + v_{AY}\hat{J} + v_{B/A-x}\,\hat{i} + v_{B/A-y}\,\hat{j} + r_{B/A-x}\dot{\hat{i}} + r_{B/A-y}\dot{\hat{j}}$$

Now we need to find expressions for $\dot{\hat{i}}$ and $\dot{\hat{j}}$. Since they are unit vectors, they do not change length; they only change direction. They are fixed to the rotating slab AB, so their changing depends solely on the slab rotation. The figure below shows this rotation in a magnified view.
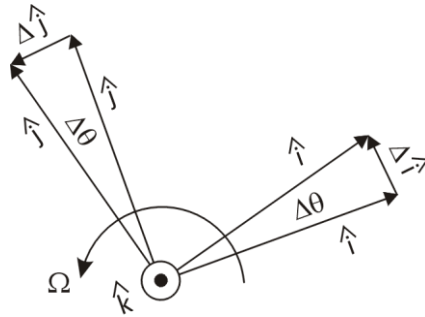
Rotation of unit vectors

After a short Δt, the slab has turned a small angle $\Delta\theta$ counter-clockwise. The change in the unit vector $\hat{\imath}$ is shown as $\Delta\hat{\imath}$. The change in $\hat{\jmath}$ is shown as $\Delta\hat{\jmath}$. As Δt is made shorter and shorter, $\Delta\hat{\imath}$ becomes perpendicular to $\hat{\imath}$, that is, rotated 90° counter-clockwise to $\hat{\imath}$, thus in the $\hat{\jmath}$-direction. $\Delta\hat{\jmath}$ is directed in the $-\hat{\imath}$-direction. In the limit, the magnitude of $\Delta\hat{\imath}$ approaches the arc length. Since $\hat{\imath}$ is a unit vector, the arc length is $1*\Delta\theta$. So $\Delta\hat{\imath} = \Delta\theta\,\hat{\jmath}$. From this, in the limit,

$$\dot{\hat{\imath}} = \frac{\Delta\hat{\imath}}{\Delta t} = \frac{\Delta\theta}{\Delta t}\,\hat{\jmath} = \Omega\,\hat{\jmath}$$

And

$$\dot{\hat{\jmath}} = \frac{\Delta\hat{\jmath}}{\Delta t} = \frac{\Delta\theta}{\Delta t}(-\hat{\imath}) = -\Omega\,\hat{\imath}$$

Now we can substitute these into the equation for $\vec{v}_B$.

$$\vec{v}_B = v_{AX}\hat{I} + v_{AY}\hat{J} + v_{B/A-x}\,\hat{\imath} + v_{B/A-y}\,\hat{\jmath} + r_{B/A-x}\,\Omega\hat{\jmath} - r_{B/A-y}\,\Omega\hat{\imath}$$

Reorganizing

$$\vec{v}_B = v_{AX}\hat{I} + v_{AY}\hat{J} + \left(v_{B/A-x} - r_{B/A-y}\,\Omega\right)\hat{\imath} + \left(v_{B/A-y} + r_{B/A-x}\,\Omega\right)\hat{\jmath}$$

On the right-hand side, after the first two terms, all of the terms have "*B/A*" in them. This indicates that the location and velocity of *B* is expressed in the little-letter coordinate system. Since we already have a lower-case *x* or *y* in the subscript, the "*B/A*" is somewhat redundant, and we shall dispense with it. Thus

$$\vec{v}_B = v_{AX}\hat{I} + v_{AY}\hat{J} + \left(v_x - r_y\,\Omega\right)\hat{\imath} + \left(v_y + r_x\,\Omega\right)\hat{\jmath}$$

Note that there are two velocities in the little-letter system. In the term $v_x - r_y\,\Omega$, for example, v_x is the rate of lengthening of the $\vec{r}_x$ vector, while $r_x\,\Omega$ is a tangential velocity due purely to the rotation of the slab. Note that even if *B* were fixed to the plate, so that $v_x = 0$, *B* still has a velocity due to the rotation of the plate and the location of *B* away from the *xy*-origin.

We can use this equation to reconstruct a vector version of $\vec{v}_B$ that does not include unit vectors.

$$\vec{v}_B = \vec{v}_A + \vec{v}_{xy} + \vec{\Omega} \times \vec{r}_{xy}$$

Since $\vec{\Omega} \times \vec{r}_{xy}$ is due to Ω, we can call this velocity $\vec{v}_\Omega$. Then

$$\vec{v}_B = \vec{v}_A + \vec{v}_{xy} + \vec{v}_\Omega$$

This is the *three-term velocity equation*. Recall that the subscript xy is a shorthand for B/A. If B is a fixed point on a component, then $v_{xy} = 0$. The term $\vec{\Omega} \times \vec{r}_{xy}$ is thus $\vec{v}_{B/A}$. Thus the derived formula is the well-known relative velocity formulae

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$$

$$\vec{v}_{B/A} = \vec{\omega}_{AB} \times \vec{r}_{B/A}$$

We've seen so far how the three-term velocity equation results from vector differentiation. A physical picture of what is at play is shown in the figure below.
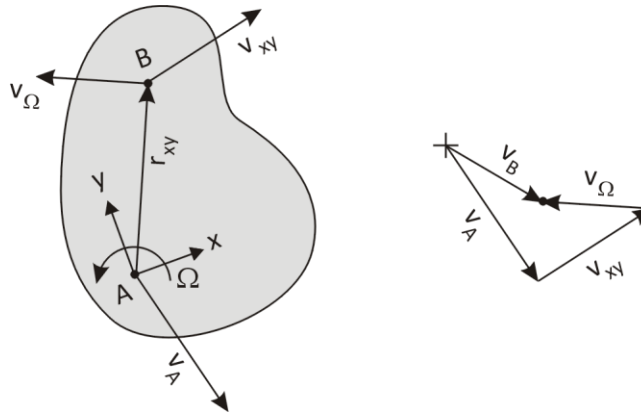
Three velocities that result from differentiating displacement vector

$\vec{v}_A$ is just the translational movement of the *xy*-origin in the *XY*-frame. $\vec{v}_{xy}$ is the translational motion of point B in the *xy*-frame, that is, on the slab. $\vec{v}_\Omega$ is the motion of B due to the rotation of the slab.

Acceleration

By definition

$$\vec{a}_B = \dot{\vec{v}}_B$$

We start with the three-term velocity equation in its detailed form above, so that we can see what accelerations result from what differentiations.

$$\vec{v}_B = v_{AX}\hat{I} + v_{AY}\hat{J} + \left(v_{B/A-x} - r_{B/A-y}\,\Omega\right)\hat{\imath} + \left(v_{B/A-y} + r_{B/A-x}\,\Omega\right)\hat{\jmath}$$

$$\dot{\vec{v}}_B = \dot{v}_{AX}\hat{I} + v_{AX}\dot{\hat{I}} + \dot{v}_{AY}\hat{J} + v_{AY}\dot{\hat{J}} + \left(\dot{v}_x - \dot{r}_y\,\Omega - r_y\,\dot{\Omega}\right)\hat{\imath} + \left(v_x - r_y\,\Omega\right)\dot{\hat{\imath}} + \left(\dot{v}_y + \dot{r}_x\,\Omega + r_x\,\dot{\Omega}\right)\hat{\jmath}$$
$$+ \left(v_y + r_x\,\Omega\right)\dot{\hat{\jmath}}$$

Again $\hat{I}$ and $\hat{J}$ belong to the fixed frame, so they don't change and are 0. Also we can substitute in the results for $\dot{\hat{\imath}}$ and $\dot{\hat{\jmath}}$ previously calculated. $\dot{r}$ is v, and $\dot{v}$ is a.

$$\dot{\vec{v}}_B = a_{AX}\hat{I} + a_{AY}\hat{J} + (a_x - v_y\,\Omega - r_y\,\dot{\Omega})\,\hat{\imath} + (v_x - r_y\,\Omega)\Omega\,\hat{\jmath} + (a_y + v_x\,\Omega + r_x\,\dot{\Omega})\,\hat{\jmath}$$
$$+ (v_y + r_x\,\Omega)(-\Omega\,\hat{\imath})$$

$$\vec{a}_B = a_{AX}\hat{I} + a_{AY}\hat{J} + (a_x - 2v_y\,\Omega - r_y\,\dot{\Omega} - r_x\,\Omega^2)\,\hat{\imath} + (a_y + 2v_x\,\Omega + r_x\,\dot{\Omega} - r_y\,\Omega^2)\,\hat{\jmath}$$

This is the *five-term acceleration equation*. The five accelerations, though they arose simply from applying the product rule to vector differentiation, can be ascribed definitive physical meaning. Let's define them first with a shorthand notation and then discuss their characteristics.

1. $\vec{a}_A = a_{AX}\,\hat{I} + a_{AY}\,\hat{J}$, the acceleration of point A in the fixed XY-frame
2. $\vec{a}_{xy} = a_x\,\hat{\imath} + a_y\,\hat{\jmath}$, the acceleration of B within the moving, rotating xy-frame
3. $\vec{a}_C = -2v_y\,\Omega\,\hat{\imath} + 2v_x\,\Omega\,\hat{\jmath}$, the Coriolis acceleration of B
4. $\vec{a}_t = -r_y\,\dot{\Omega}\,\hat{\imath} + r_x\,\dot{\Omega}\,\hat{\jmath}$, the tangential acceleration of B
5. $\vec{a}_n = -r_x\,\Omega^2\,\hat{\imath} - r_y\,\Omega^2\,\hat{\jmath}$, the normal acceleration of B

So

$$\vec{a}_B = \vec{a}_A + \vec{a}_{xy} + \vec{a}_C + \vec{a}_t + \vec{a}_n$$

$\vec{a}_A$ and $\vec{a}_{xy}$ are both independent accelerations and do not involve the rotation of the slab. Normally in an analysis they will have to be given or observed. The other three accelerations do involve the rotation of the slab. $\vec{a}_t$ involves a speed-up of the tangential velocity, $\vec{v}_\Omega$, simply because the spin speed is increasing…or a slow-down in $\vec{v}_\Omega$ because the spin speed is decreasing. $\vec{a}_n$ is due to the constant change in direction of $\vec{v}_\Omega$ as it seemingly rotates about A. Note that this is directed opposite to $\vec{r}_{xy}$, which is $\vec{r}_{B/A}$. The third term, Coriolis acceleration, is the most puzzling and misunderstood of the five. It results from a combination of movements: movement on the slab while it is rotating. It was not even known until long after Newton developed his three laws and Euler extended them to rigid bodies. I have written a separate article on Coriolis acceleration that can be found at www.aoengr.com/Dynamics/CoriolisAcceleration.pdf.

The three accelerations involving $\vec{\Omega}$ can also be cast as cross products.

3. $\vec{a}_C = 2\,\vec{\Omega} \times \vec{v}_{xy}$, the Coriolis acceleration of B
4. $\vec{a}_t = \dot{\vec{\Omega}} \times \vec{r}_{xy}$, the tangential acceleration of B
5. $\vec{a}_n = \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_{xy})$, the normal acceleration of B

That these are equivalent to the unit-vector versions above can be seen as follows. The magnitude equivalency can be verified by inspection. The directions can be seen in that in 2-D, positive $\vec{\Omega}$ and $\dot{\vec{\Omega}}$ point out of the slab. Pre-crossing them with a vector produces a vector that is rotated 90° to the right of the second vector in the cross-product. Also, since both $\vec{\Omega}$ and $\dot{\vec{\Omega}}$ point in the $\hat{k}$ direction (if they are positive), they will convert $\hat{\imath}$-components into $\hat{\jmath}$-components and $\hat{\jmath}$-components into $-\hat{\imath}$-

components. Note that $\vec{a}_c$ and $\vec{a}_t$ have y-components in the $-\hat{\imath}$-direction and x-components in the $\hat{\jmath}$-direction. With $\vec{a}_n$, there is a double pre-crossing by $\vec{\Omega}$. Thus the direction is turned 180° around from $\vec{r}_{xy}$. In the second pre-cross, the x-components that had been turned into the $\hat{\jmath}$-direction get turned another 90° into the $-\hat{\imath}$-direction. Note that in the unit-vector version of $\vec{a}_n$, x-components stay as $\hat{\imath}$-components, and y-components stay as $\hat{\jmath}$-components...but both are now negative.
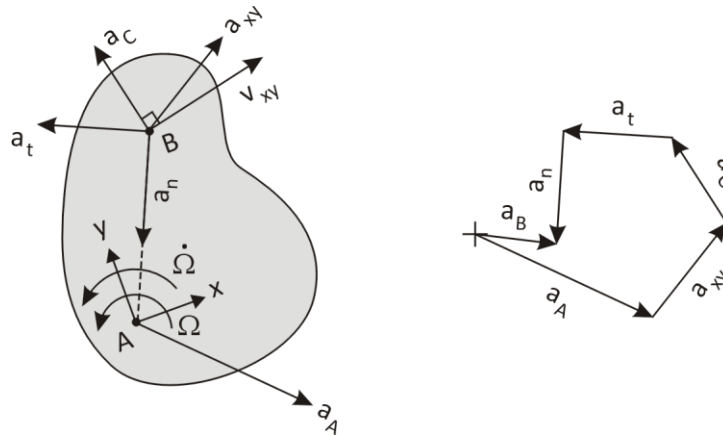

Five accelerations with a rotating frame

The figure above shows an example of the five accelerations. $\vec{a}_A$ and $\vec{a}_{xy}$ are independent and could be drawn in any direction. The Coriolis acceleration is perpendicular and to the left of $\vec{v}_{xy}$ since the slab is rotating counter-clockwise. $\vec{a}_t$ is perpendicular and to the left of the line from A to B since $\dot{\vec{\Omega}}$ is counter-clockwise. $\vec{a}_n$ leads from B back toward the apparent center of rotation at A.

Though this is explained in 2-D for visualization purposed, it is equally valid for 3-D.

Also noteworthy is that the center of rotation of the slab does not have to be at A. It does look to an observer at A as if the slab is rotating about him. But the rotation point does not even have to be fixed. It could change with time, like an instant center does, in the general case. It could even be, if we take the special case of B not moving on the slab, that B is the center of rotation. But all of the above is still true, and we can center our rotating coordinate system at any point on the slab. I would like to work out this counter-case at some point, just to demonstrate it.

Relationship between large- and small-letter coordinate systems

In the three-term velocity equation and in the five-term acceleration equation, the first term on the right will be in the large-letter coordinate system and the rest in the small-letter coordinate system. It will be desired to get everything in the large-letter system, so expressions need to be worked out for $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ in terms of $\hat{I}$, $\hat{J}$, and $\hat{K}$. We return to the original drawing, which explained the scenario, and focus on the two sets of unit vectors.
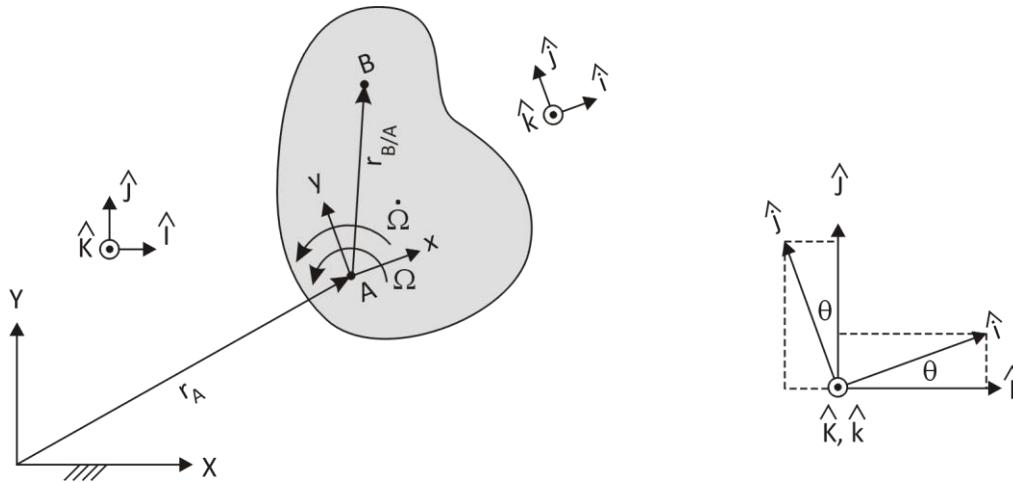
Relationship between small- and large-letter unit vectors

Referring to the magnified drawing of the unit vectors at right, we can see that

$$\hat{\imath} = \cos\theta\,\hat{I} + \sin\theta\,\hat{J}$$
$$\hat{\jmath} = -\sin\theta\,\hat{I} + \cos\theta\,\hat{J}$$

The last four terms of the acceleration are written in terms of $\hat{\imath}$ and $\hat{\jmath}$. They will have to have these expressions substituted in to have these acceleration in the fixed coordinate system.